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To explore the concept of stability in an age-structured population with migration, a Markov transition matrix model is built, where age classes can be of different length, and the time step is not necessarily equal to the length of the age class. The conditions under which a vector of the model has a steady population structure are identified, as well as those under which the age structure converges to a given steady state, through a series of decisions or controls of letting immigrants in or forbidding them entry into the country. The decisions are expressed as vectors of proportions of immigrants. In the steady state, when the increment of population is proportional to its size, the age- or stage-structure remains unchanged between transitions.

Keywords: age-structured population; matrix model; stability; target-control

1. INTRODUCTION

Immigration in age-structured populations was mainly investigated in continuous time (Ackleh and Deng, 2005; Chen, 1988; He, Wang and Ma, 2004; Norhayati and Wake, 2003; Skakauskas, 2004) and less frequently in discrete-time (Caswell, 2001; Jensen, 1997; Levin,1998; Logofet and Klochkova, 2002; Pollard, 1973; Rogers, 1968; Wikan, 2004). Here, we consider a population open to migration and use

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Markov chain theory. We avoid the assumptions that age classes have equal length and that stages should be of equal duration. We explore the existence of a sequence of controls on immigration to reach a target of a given age- and stage-structure using a matrix model with migration.

2. POPULATION DYNAMICS WITH MIGRATION

2.1. Definition of Variables

The population is divided into k age classes, not necessarily equal in length, and stages ($\mathrm{t} = 0,\ 1,\ 2, \ldots$) not necessarily of the length of the age classes. We need:

1. The age-structured population $x(t) = (x_1(t),\ x_2(t), \ldots, x_k(t))$, where $x_i(t)$ is the population size of age class i, $N(t) = \sum_{i=1}^{k} x_i(t)$ is the population size.
2. The age-structure $p(t) = (p_1(t), p_2(t), \ldots, p_k(t))$ with $p_i(t) = x_i(t)/N(t),\ p_i(t) \geq 0, \sum_{i=1}^{k} p_i(t) = 1$.
3. The semi-transition matrix

$$Q = (p_{ij})_{k\times k} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix}$$

   where p_{ij} is the proportion of individuals moving from age class i to age class j. The p_{ij}s are either transition probabilities or reproductive outputs, including the reproduction of the native populations and individuals in age class i surviving and remaining in age class i or ageing into age class j. $\boldsymbol{Q}$ is not a stochastic matrix, taking more general forms and not constrained by $\sum_{j=1}^{k} p_{ij} = 1,\ i = 1, 2, \ldots, k$.
4. The emigration structure $\boldsymbol{e} = (e_1, e_2, \cdots, e_k)$, where e_i is the proportion in age class i of the population emigrated from or died in age class i at every stage. An estimate is the average value over a past period. The total population emigrated from or died at stage t is $\mathrm{W(t)} = \sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{e_i x_i(t)} = \mathrm{x(t)e^T}$.
5. The vector $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ of proportions r_i of immigrants in age class i whatever the stage attained. It is estimated as the average value over a past period. $r_i R(t)$ is the total number of individuals immigrating into age class i at stage t, where $R(t)$ is the total number of immigrants at stage t.

From this definition $\mathrm{p_{ij}}, e_i,$ and r_i satisfy

$$\mathrm{p_{ij}} \geq 0,\ \mathrm{e_i} \geq 0, \quad \sum_{\mathrm{j=1}}^{\mathrm{k}} \mathrm{p_{ij}} + \mathrm{e_i} = 1 \tag{1}$$

$$\mathrm{r_i} \geq 0, \qquad \sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{r_i} = 1. \tag{2}$$

The transition between age classes is represented on Figure 1.

2.2. The Model

The matrix model of the age-structured population is given by the recursion in *N(t)* and $x_i(t)$:

$$\mathrm{N(t+1) = N(t) + R(t) - W(t)} \tag{3}$$

$$x_j(t+1) = \sum_{i=1}^{k} p_{ij} x_i(t) + r_j R(t)$$
$$\mathrm{j = 1, 2, \ldots, k.} \tag{4}$$

Expressed in a vector form, Eq. (4) becomes

$$\mathbf{x}(\mathrm{t}+1) = \mathbf{x}(\mathrm{t})\mathbf{Q} + \mathrm{R(t)}\mathbf{r} \tag{5}$$

Let $M(t) = N(t+1) - N(t)$, the increment of the total number of individuals in the stage from t to $t+1$. From Eq. (3) and $W(t) = x(t)e^T$, we obtain

$$\mathrm{R(t) = W(t) + M(t) = x(t)}\mathbf{e}^\mathrm{T} + \mathrm{M(t)}. \tag{6}$$

Replacing *R(t)* of Eq. (5) with Eq. (6) and simplifying Eq. (5), we get the recursion

$$\mathbf{x}(\mathrm{t}+1) = \mathbf{x}(\mathrm{t})(\mathbf{Q} + \mathbf{e}^\mathrm{T}\mathbf{r}) + \mathrm{M(t)}\mathbf{r} \tag{7}$$

Consider the matrix $\mathbf{P} = \mathbf{Q} + \mathbf{e}^\mathrm{T}\mathbf{r}$. From Eqs. (1) and (2), the sum of the rows of $\mathbf{P}$ equals the vector *1*. Hence $\mathbf{P}$ is stochastic, which is a condition

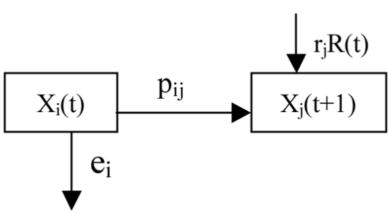

FIGURE 1 Transition between age classes with migration.

of the Markov chain. Eq. (7) is written as

$$\mathbf{x}(\mathrm{t}+1) = \mathbf{x}(\mathrm{t})\mathbf{P} + \mathrm{M}(\mathrm{t})\mathrm{r}. \tag{8}$$

Eqs. (5) and (8) are matrix population models with in- and out-migration. With the transition matrix $\mathbf{Q}$, the vector $\mathbf{r}$ of proportions of immigrants, the initial age-structured population $\mathbf{x}(0)$, the total number *R(t)* of immigrants, or the increment *M(t)* of population size, we obtain the changes of $\mathbf{x}(\mathrm{t})$ by Eq. (5) or Eq. (8), respectively.

Case I: When the total number of individuals at every stage increases of a fixed percent α, or $M(t) = \alpha N(t)$, we get $N(t+1) = (1+\alpha)N(t)$. Replacing $\mathbf{x}(\mathrm{t})$ in Eq. (8) by *p(t)*, we rewrite Eq. (8) as $p(t+1) = (1+\alpha)^{-1}(p(t)\mathbf{P} + \alpha\mathbf{r})$.

Case II: If out-migration counterbalances in-migration, the population size N(t) should be unchanged, or $\mathrm{M}(\mathrm{t}) = 0$ (or $\alpha = 0$). Eq. (8) is reduced to the simple recursion

$$\mathrm{p}(\mathrm{t}+1) = \mathrm{p}(\mathrm{t})\mathbf{P} \tag{9}$$

where $\mathbf{P} = \mathbf{Q} + \mathbf{e}^{\mathrm{T}}\mathbf{r}$.

2.3. Results

For the Markov chain (9), which is regular (Bartholomew, 1973; Gourley and Wu, 2004), the first right hand eigenvector of $\mathbf{P}$ is an age structure $\mathbf{a} = (\mathrm{a}_1, \ldots, \mathrm{a}_\mathrm{k})$ satisfying

$$\mathbf{a} = \mathbf{aP}, \tag{10}$$

The vector $\mathbf{a}$ is the steady population structure.

Proposition 1: $\mathbf{a}$ is a steady population structure if and only if $\mathbf{a} \geqq \mathbf{aQ}$ where $\mathrm{Q} = (\mathrm{p}_{\mathrm{ij}})_{\mathrm{k}\times\mathrm{k}}$ is a semi-transition matrix.

Proof. From Eq. (10) and $\mathbf{P} = \mathbf{Q} + \mathrm{e}^T\mathbf{r}$,

$$r = (a - aQ)/ae^T, \qquad \mathbf{ae}^{\mathrm{T}} > 0. \tag{11}$$

$\mathbf{r}$ satisfies $\sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{r}_\mathrm{i} = 1$
From Eq. (1):

$$\begin{aligned}\sum_{j=1}^{k} r_j &= \left(\sum_{j=1}^{k} a_j - \sum_{j=1}^{k}\sum_{i=1}^{k} a_i p_{ij}\right)/ae^T \\ &= \left(\sum_{j=1}^{k} a_j - \sum_{i=1}^{k} a_i \sum_{j=1}^{k} p_{ij}\right)/ae^T \\ &= \left(\sum\nolimits_{j=1}^{k} a_j - \sum\nolimits_{i=1}^{k} a_i(1-e_i)\right)/ae^T = 1\end{aligned}$$

From Eq. (2), $\mathbf{r}$ still needs to satisfy $r_i \geq 0$. From (11), $\mathbf{a}$ is the steady population structure if and only if $\mathbf{a} \geqq \mathbf{aQ}$.

Proposition 2: $\mathbf{a}$ is a steady population structure if and only if $a = \sum_{i=1}^{k} b_i s_i$, where $b_i \geq 0$, $\sum_{i=1}^{k} b_i = 1, s_i$ is a vector, $\mathbf{s}_i = m_i/u_i$, $\mathbf{m}_i$ is the ith row vector of $(\mathbf{I}–\mathbf{Q})^{-1}$, u_i is the sum of the elements of $\mathbf{m}_i$

Proof: From Eq. (1), $\sum_{j=1}^{k} p_{ij} < 1$ so that $(\mathbf{I}–\mathbf{Q})$ is invertible, $\mathbf{I}$ is the unit matrix.

Take $\mathbf{M} = (m_{ij})_{k \times k} = (\mathbf{I}–\mathbf{Q})^{-1}$. From Eq. (10), $\mathbf{a}$ is a steady population structure if and only if

$$\mathbf{a} = \mathbf{ae}^{T}\mathbf{rM} \tag{12}$$

Denote $\mathbf{m}_i = (m_{i1}, m_{i2}, \ldots, m_{ik})$ and $u_i = \sum_{j=1}^{k} m_{ij}$. Then $rM = \sum_{i=1}^{k} r_i m_i$.

From Eq. (12), $a1^T = \sum_{i=1}^{k} a_i = 1$, $\mathbf{m}_i\mathbf{1}^T = u_i$, $1 = (1, 1, \ldots, 1)$ and

$$ae^T = \frac{1}{\sum_{i=1}^{k} r_i u_i} \tag{13}$$

Hence Eq. (12) is rewritten as $\mathbf{a} = \sum_{i=1}^{k} (r_i m_i / \sum_{j=1}^{k} r_j u_j)$. Denote $b_i = r_i u_i / \sum_{j=1}^{k} r_j u_j$ and $s_i = m_i/u_i$. Then $\mathbf{a} = \sum_{i=1}^{k} b_i s_i$, where $\sum_{i=1}^{k} b_i = 1$, $M = (I - Q)^{-1} = I + Q + Q^2 + \cdots$ is a non-negative matrix and the elements m_{ij} of M are also non-negative, which implies that u_i is nonnegative. In addition, from Eq. (13) $\sum_{i=1}^{k} r_i u_i = (ae^T)^{-1} > 0$. So $r_i > 0$ if and only if $b_i \geqq 0$.

The s_i are steady population structures, too. Given the transition matrix $\mathbf{Q}$, we get the range and the expression of the steady population structure $\mathbf{a}$ by Propositions 1 and 2, respectively.

3. CONVERGENCE OF THE AGE-STRUCTURED POPULATION UNDER A SEQUENCE OF CONTROLS ON A VECTOR OF PROPORTIONS OF IMMIGRANTS

Assume that the age-structured population is open to migration. The vector $\mathbf{r}(t) = (r_1(t), r_2(t), \ldots, r_k(t))$ of proportions of immigrants is a function of stage t. Assume that the vector $\mathbf{r}(t)$ of proportions of immigrants can be controlled so as to have the age-structure converging to a steady structure $\mathbf{a}^*$. We shall find such controls.

Theorem: Given the initial age-structure $\mathbf{x}(1)$, the vector $\mathbf{e}$ of proportions of emigrants and the semi-transition matrix $\mathbf{Q}$, there exists a sequence of vectors of proportions of immigrants $\{\mathbf{r}^*(\mathrm{t})\}$ such that the population open to migration converges to a steady structure $\mathbf{a}^*$.

Proof. Consider a steady structure $\mathbf{a}^*$. From Eq. (10) there is a vector $\mathbf{r}_0$ of proportions of immigrants such that

$$\mathbf{a}^* = \mathbf{a}^*(\mathbf{Q} + \mathbf{w}^{\mathrm{T}}\mathbf{r}_0) = \mathbf{a}^*\mathbf{P}_0$$

Using dynamic programming, an vector $\mathbf{r}^*(\mathrm{t})$ is obtained such that $\|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\mathrm{P}^*(\mathrm{t})\| \leq \|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\mathrm{P}_0\|$ (Zhang, 1998). With the age-structure $\mathbf{a}^*(\mathrm{t}+1)$ at the $(\mathrm{t}+1)$th stage obtained after moving between stages and after immigration,

$$\mathbf{a}^*(\mathrm{t}+1) = \mathbf{a}^*(\mathrm{t})\mathbf{P}^*(\mathrm{t})$$

where $\mathbf{P}^*(\mathrm{t}) = \mathbf{Q} + \mathbf{w}^{\mathrm{T}}\mathbf{r}^*(\mathrm{t})$, $\mathbf{r}^*(\mathrm{t})$ is the vector of proportions of immigrants. Zhang (1998) proved that if $\mathbf{A} = (\mathrm{a_{ij}})_{\mathrm{n}\times\mathrm{n}}$ is a complex matrix, $|\mathbf{A}| \leq \max_i \sum_j |a_{ij}|$ and $|\mathbf{A}| \leq \max_j \sum_i |a_{ij}|$, where $|\mathbf{A}| = \max\{|\lambda|\}$ and where the λs are the eigenvalues of the matrix. $|\mathbf{A}|$ is the spectral radius of the matrix $\mathbf{A}$. From the definition of $\mathbf{r}^*(\mathrm{t})$, a lemma in the Appendix, and the fact that the matrix $\mathbf{P}_0$ is stochastic,

$$\begin{aligned}
\|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t}+1)\| &= \|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\mathrm{P}^*(\mathrm{t})\| \\
&\leq \|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\mathrm{P}_0\| \\
&= \|(\mathrm{a}^* - \mathrm{a}^*(\mathrm{t}))\mathrm{P}_0\| \\
&\leq \|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\||\mathrm{P}_0| \\
&\leq \|\mathrm{a}^* - \mathrm{a}^*(\mathrm{t})\|
\end{aligned}$$

for any stage t. The population structure $\mathbf{a}^*(\mathrm{t})$ converges to the steady population structure $\mathbf{a}^*$ under a sequence of controlled vectors $\{\mathbf{r}^*(\mathrm{t})\}$ of proportions of immigrants.

4. CONCLUSION

The determination of the elements p_{ij} of the semi-transition matrix $\mathbf{Q}$ plays an important role in prediction. p_{ij} includes survival probabilities and birth rates and should be a function of t (Davydova, Diekmann and van Gils, 2003; Grant and Benton, 2000; Takada and Nakajima, 1998). The elements of matrix $\mathbf{Q}$ depend on the contrast between the duration of observation and the lengths of the age classes (Cushing, 1998; Behncke, 2000; Hiebeler, 1998; Neubert and Caswell, 2000). If p_{ij} are constant and the increment of population is proportional to its size, the discrete-time matrix model describes the dynamics of an age-structured population in case *I* and *II*. Propositions 1 and 2 give the necessary and sufficient conditions that a vector of the model has a steady population structure. The expression and range of the steady population structure are also provided. We prove that, for case I, there is a sequence of controls on immigration letting the population converge to a given age- or stage- structure, a steady state which satisfies the conditions of Propositions 1 and 2.

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APPENDIX

Lemma 1. Let $\lambda_1 \ldots \lambda_n$ be the eigenvalues of the square matrix **A,** $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$. Then

$$\|\mathbf{Ax}\| \le |\mathbf{A}|\|\mathbf{x}\|$$

where $\|\mathbf{x}\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ is the norm of the vector $\mathbf{x}$, the spectral radius of the matrix **A** is $|\mathbf{A}| = \max(|\lambda_i|)$ with the λ_is the eigenvalues of the matrix **A**.

Proof. The eigenvalues of $\mathbf{A}^T\mathbf{A}$ are $\lambda_1^2, \lambda_2^2, \ldots \lambda_n^2$. $\mathbf{A}^T\mathbf{A}$ is symmetric, then there exists an orthogonal matrix $\mathbf{P}$, $\mathbf{P}^{-1} = \mathbf{P}^T$ such that

$$\mathbf{P}^{-1}\mathbf{A}^T\mathbf{AP} = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix}$$

or

$$\mathbf{A}^T\mathbf{A} = \mathbf{Q}^T \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} \mathbf{Q}$$

where $\mathbf{Q} = \mathbf{P}^{-1}$. Consider $\mathbf{y} = \mathbf{Qx} = (\mathrm{y}_1, \mathrm{y}_2, \ldots\ldots, \mathrm{y}_\mathrm{n})^\mathrm{T}$, then

$$\|\mathbf{Ax}\|^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = \mathbf{y}^T \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{bmatrix} \mathbf{y}.$$

Because $\mathbf{Q}$ is orthogonal, $\|\mathbf{x}\| = \|\mathbf{y}\|$, and

$$\|\mathbf{Ax}\| = \sqrt{\lambda_1^2 y_1 + \lambda_2^2 y_2 + \cdots + \lambda_n^2 y_n} \leq |\mathbf{A}|\|\mathbf{y}\|$$

so that

$$\|\mathbf{Ax}\| \leq |\mathbf{A}|\|\mathbf{x}\|.$$

Because the transposed matrix $\mathbf{A}^\mathrm{T}$ has the same eigenvalues as $\mathbf{A}$ and thanks to lemma 1,

$$\|\mathbf{x}^\mathrm{T}\mathbf{A}\| \leq \|\mathbf{x}^\mathbf{T}\||\mathbf{A}|.$$